The Nonlinear Schrödinger Equation on Metric Graphs

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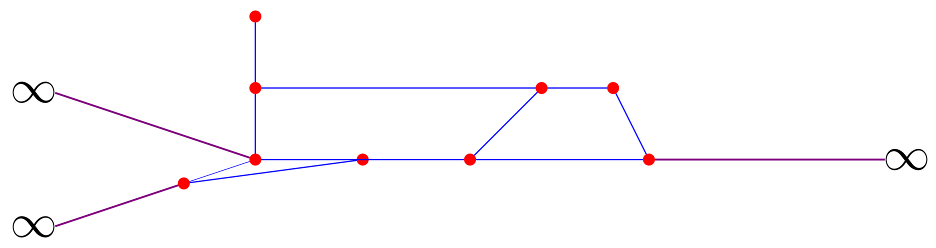
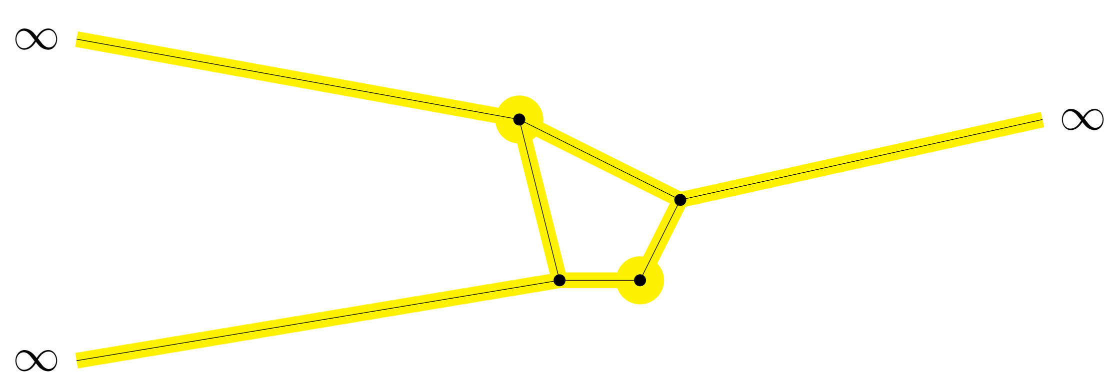
What is a metric graph?

A metric graph is made of vertices and of edges joining the vertices or going to infinity.



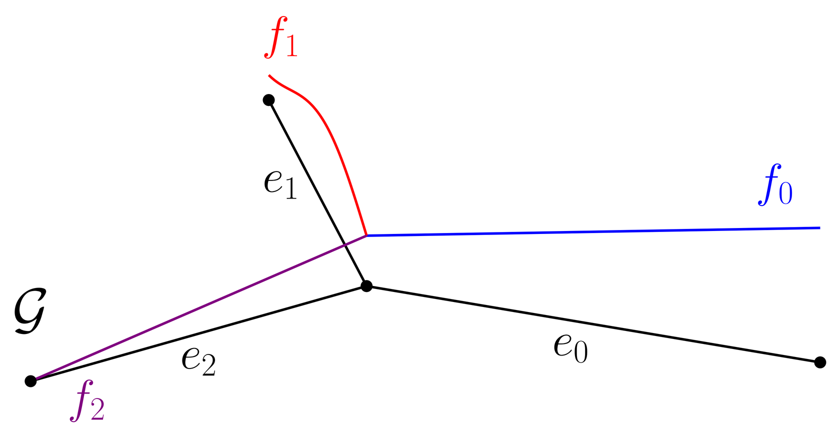
- *Metric* graphs: the lengths of edges are important.
- The edges going to infinity are halflines and have *infinite length*.
- A metric graph is *compact* if and only if it has a finite number of edges of finite length.

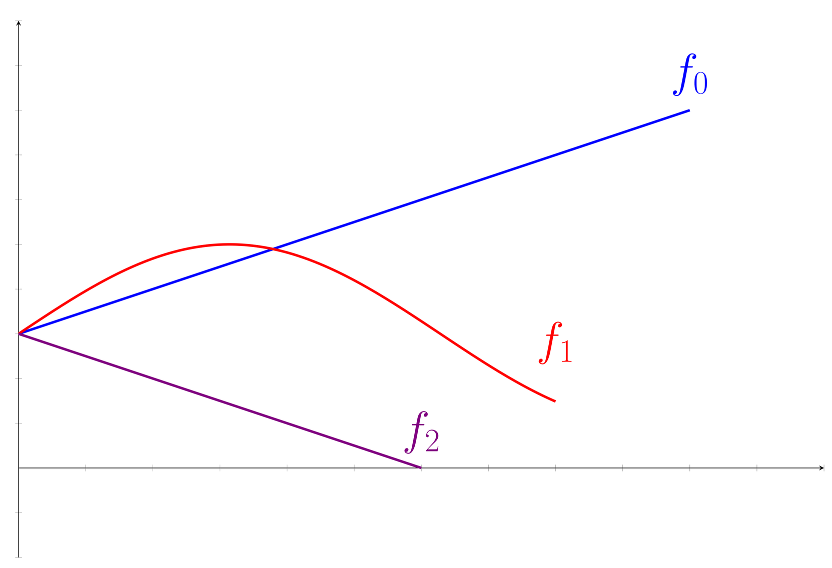
Metric graphs may be used to model structures where *only one spatial direction is important*.



Functions on metric graphs

Here is an example of a metric graph $\mathcal{G}$ with three edges e_0 (length 5), e_1 (length 4) and e_2 (length 3), a function $f : \mathcal{G} \to \mathbb{R}$, and the three associated real functions f_0, f_1 and f_2.





One may naturally perform operations over functions defined on metric graphs, such as integration:

$$\int_{\mathcal{G}} f \,\mathrm{d}x := \int_0^5 f_0(x)\,\mathrm{d}x + \int_0^4 f_1(x)\,\mathrm{d}x + \int_0^3 f_2(x)\,\mathrm{d}x$$

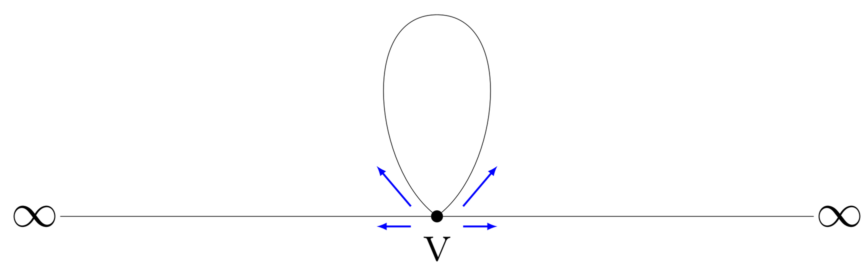
The NLS equation on graphs

Given constants $p > 2$ and $\lambda > 0$, we are interested in solutions $u \in L^2(\mathcal{G})$ of

$$\begin{cases} -u'' + \lambda u = |u|^{p-2}u & \text{on each edge } e \text{ of } \mathcal{G}, \\ u \text{ is continuous} & \text{for every vertex } \mathrm{v} \text{ of } \mathcal{G}, \\ \displaystyle\sum_{e \succ \mathrm{v}} \frac{\mathrm{d}u}{\mathrm{d}x_e}(\mathrm{v}) = 0 & \text{for every vertex } \mathrm{v} \text{ of } \mathcal{G}. \end{cases} \tag{NLS}$$

Kirchhoff's condition

In (NLS), $e \succ \mathrm{v}$ means the sum ranges over all edges of vertex v and $\frac{\mathrm{d}u}{\mathrm{d}x_e}(\mathrm{v})$ is the outgoing derivative of u at v.



Notation: We denote by $\mathcal{S}_\lambda(\mathcal{G})$ the set of nonzero solutions.

Variational formulation

We work on the Sobolev space

$$H^1(\mathcal{G}) := \left\{ u : \mathcal{G} \to \mathbb{R} \mid u \text{ is continuous}, u, u' \in L^2(\mathcal{G}) \right\}.$$

The solutions of (NLS) are the critical points of the *action functional*

$$J_\lambda(u) := \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} + \frac{\lambda}{2}\|u\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\|u\|^p_{L^p(\mathcal{G})}.$$

It is not bounded from below on $H^1(\mathcal{G})$, since if $u \neq 0$ then

$$J_\lambda(tu) = \frac{t^2}{2}\|u'\|^2_{L^2(\mathcal{G})} + \frac{\lambda t^2}{2}\|u\|^2_{L^2(\mathcal{G})} - \frac{t^p}{p}\|u\|^p_{L^p(\mathcal{G})} \xrightarrow[t \to +\infty]{} -\infty.$$

A common strategy is to introduce the *Nehari manifold*:

$$\begin{aligned} \mathcal{N}_\lambda(\mathcal{G}) &:= \left\{ u \in H^1(\mathcal{G}) \setminus \{0\} \mid J'_\lambda(u)[u] = 0 \right\} \\ &= \left\{ u \neq 0 \mid \|u'\|^2_{L^2(\mathcal{G})} + \lambda\|u\|^2_{L^2(\mathcal{G})} = \|u\|^p_{L^p(\mathcal{G})} \right\}. \end{aligned}$$

If $u \in \mathcal{N}_\lambda(\mathcal{G})$, then

$$J_\lambda(u) = \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^p_{L^p(\mathcal{G})}.$$

In particular, J_λ is bounded from below on $\mathcal{N}_\lambda(\mathcal{G})$.

Two action levels

We define two key quantities:

$$c_\lambda(\mathcal{G}) := \inf_{u \in \mathcal{N}_\lambda(\mathcal{G})} J_\lambda(u); \qquad \sigma_\lambda(\mathcal{G}) := \inf_{u \in \mathcal{S}_\lambda(\mathcal{G})} J_\lambda(u).$$

$c_\lambda(\mathcal{G})$ is the "ground state" action level. If this is a minimum, then $c_\lambda(\mathcal{G}) = \sigma_\lambda(\mathcal{G})$ and all minimizers are solutions, called *ground states*.

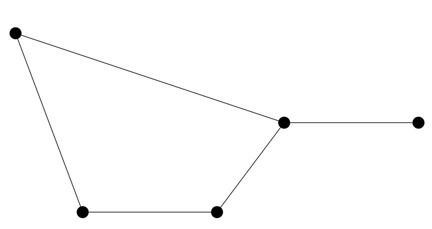
An analysis shows that four cases are possible:

A1) $c_\lambda(\mathcal{G}) = \sigma_\lambda(\mathcal{G})$ and both infima are attained;
B1) $c_\lambda(\mathcal{G}) < \sigma_\lambda(\mathcal{G})$, $\sigma_\lambda(\mathcal{G})$ is attained but not $c_\lambda(\mathcal{G})$;
A2) $c_\lambda(\mathcal{G}) = \sigma_\lambda(\mathcal{G})$ and neither infimum is attained;
B2) $c_\lambda(\mathcal{G}) < \sigma_\lambda(\mathcal{G})$ and neither infimum is attained.

Theorem (De Coster, Dovetta, G., Serra [1])
For every $p > 2$, every $\lambda > 0$, and every choice of alternative between A1, A2, B1, B2, there exists a metric graph $\mathcal{G}$ where this alternative occurs.

Examples for the four cases

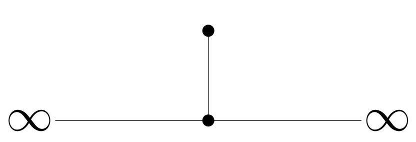
Case A1



Compact graphs

The real line

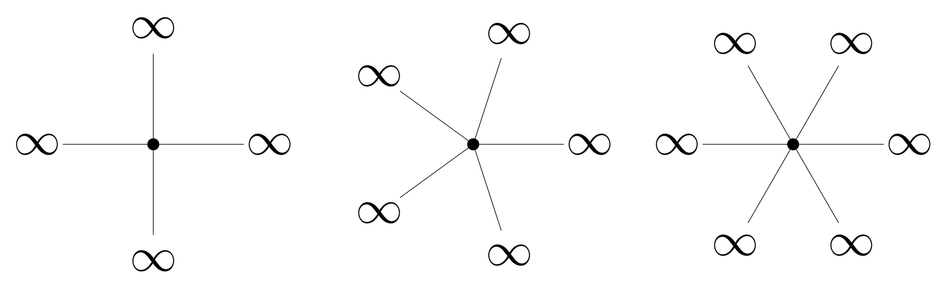
The half-line



Graphs with $c_\lambda(\mathcal{G}) < c_\lambda(\mathbb{R})$

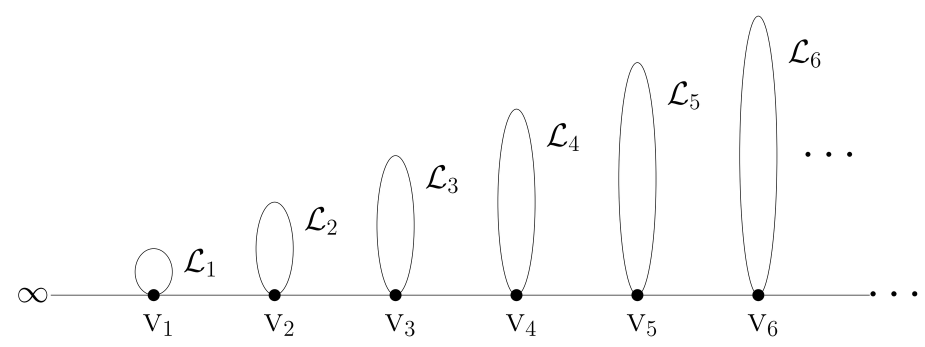
Examples (cont.)

Case B1

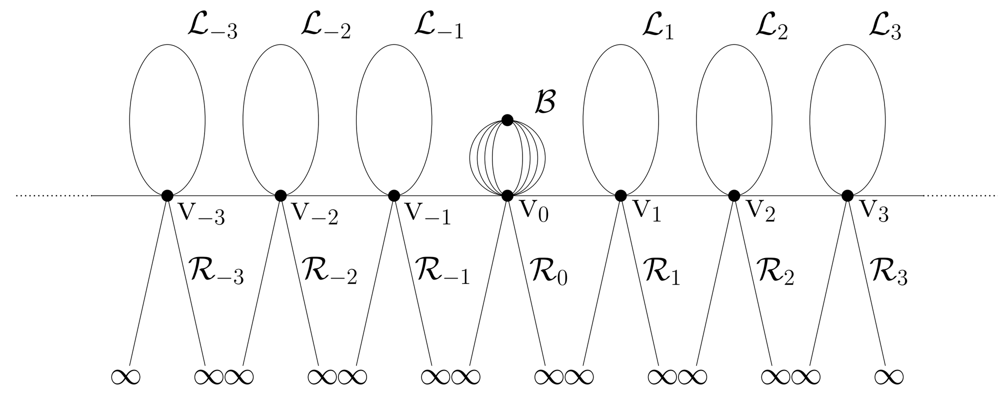


N-star graphs with $N \geq 3$ half-lines

Case A2



Case B2



Sign-changing solutions

Given a function u, we let $u^+ := \max(u, 0)$, $u^- := \min(u, 0)$ and define the *nodal Nehari set* as

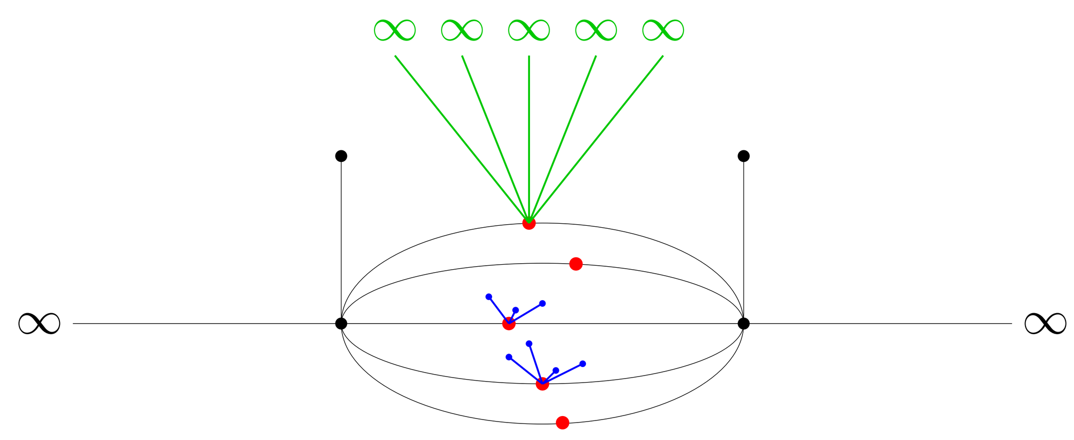
$$\mathcal{M}_\lambda(\mathcal{G}) := \left\{ u \in H^1(\mathcal{G}) \mid u^\pm \in \mathcal{N}_\lambda(\mathcal{G}) \right\}$$

The nodal Nehari set contains all nodal solutions of (NLS). We consider the minimization problem

$$\inf_{v \in \mathcal{M}_\lambda(\mathcal{G})} J_\lambda(v).$$

If this is a minimum, all minimizers are *nodal ground states*.

Theorem (De Coster, Dovetta, G., Serra, Troestler [2])
For every $k, m, n \in \mathbb{N}$ with $m \geq 2$, there exists a graph $\mathcal{G}$ and a nodal ground state u on $\mathcal{G}$ such that the set $u^{-1}(\{0\})$ is the union of k isolated points, m half-lines and n line segments.



References

[1] De Coster C., Dovetta S., Galant D., Serra E., *On the notion of ground state for nonlinear Schrödinger equations on metric graphs*, Calc. Var. **62**, 159 (2023).

[2] De Coster C., Dovetta S., Galant D., Serra E., Troestler C., *Constant sign and sign changing NLS ground states on noncompact metric graphs*, Anal. PDE **19**, no. 2, 203–240 (2026).

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